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OPTIMAL INTERDICTION OF A SUPPLY NETWORK

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ABSTRACT

Under certain conditions, the re-supply capability of a combatant force may be limited by the characteristics of the transportation network over which supplies must flow. Interdiction by an opposing force may be used to reduce the capacity of that network. The effects of such efforts vary for differing missions and targets. With only a limited total budget available, the interdictor must decide which targets to hit, and with how much effort. An algorithm is presented for determining the optimum interdiction plan for minimizing network flow capacity when the minimum capacity on an arc is positive and the cost of interdiction is a linear function of arc capacity reduction.

The problem of reducing the maximum flow in a network has received considerable interest recently [1, 3, 8, 9], primarily as a consequence of the problem of interdicting supply lines in limited warfare. In this paper an algorithm is presented for reducing the maximum flow in such a network when the resources of the interdicting force are limited. A typical problem is that of the strike planner who must determine the best way to allocate a limited number of aircraft to interdict an enemy's supply lines on a particular day.

The network is assumed to be capacity limited and to be representable as a planar connected graph of nodes and undirected capacitated arcs. Further, it is assumed to have a single source through which flow enters the network and a single sink through which flow leaves. The maximum flow through such networks is easily determined by finding the minimum cut set where a cut set is defined as a set of arcs which, when removed, causes a network to be partitioned into two subgraphs, one subgraph containing the source node and the other containing the sink node. The value of a cut set is the sum of the flow capacities of its arcs. The minimum cut set is that cut set whose value is the minimum of all cut sets of a network. The max-flow min-cut theorem states that the maximum flow possible through the network is equal to the value of the minimum cut set [4, 5].

In the interdiction problem, an arc (i, j) is assumed to have a maximum flow capacity, $u_{ij} \geq 0$, and a minimum flow capacity, $l_{ij} \geq 0$. At least one arc of the network is assumed to have $l_{ij} > 0$. As a consequence of interdiction, the actual capacity, m_{ij}, on an arc will be somewhere in the range $0 \leq l_{ij} \leq m_{ij} \leq u_{ij}$.

If we assume that the interdictor incurs a cost, C_{ij}, per unit of capacity decrease, then his total cost for reducing an arc's capacity from u_{ij} to m_{ij} will be $C_{ij}[u_{ij} - m_{ij}]$. If we assume the interdictor has a total budget limitation, K, which he cannot exceed, then

$$\sum_{\text{all } (i,j)} C_{ij}[u_{ij} - m_{ij}] \leq K.$$

The cost, C_{ij}, might represent the number of sorties required to reduce arc capacity by one unit and K might represent the total number of sorties which can be flown in a 24-hour period.

The interdictor's problem is to find a set of m_{ij} which minimizes the maximum flow in the supply network subject to

$$\sum_{\text{all }(i,j)} C_{ij}[u_{ij}-m_{ij}] \leq K$$

and

$$l_{ij} \leq m_{ij} \leq u_{ij} \qquad \text{for all } (i,j).$$

Topological Dual

In resolving the interdictor's problem we will make use of the topological dual. This dual, when defined, is another network in which the arcs have lengths instead of capacities. A one-to-one correspondence exists between the cut sets of the original or primal network and the loopless paths through the dual. The problem of finding the minimum cut set in the primal is equivalent to finding the shortest path through the dual [4].

Let the original maximum flow network be called the primal. To construct the topological dual we begin by adding an artificial arc connecting the source to the sink in the primal. The resulting network will be referred to as the modified primal and the area surrounding this network will be referred to as the external mesh. A dual is defined if and only if the modified primal is planar; a planar network being one that can be drawn on a plane such that no two arcs intersect except at a node.

When defined, a dual may be constructed for the interdiction problem in the following manner [9]:

1. Place a node in each mesh of the modified primal including the external mesh. Let the source of the dual be the node in the mesh involving the artificial arc and the sink be the node in the external mesh.
2. For each arc in the primal (except the artificial arc) construct an arc that intersects it and joins with nodes in the meshes adjacent to it.
3. Assign each arc of the dual a length equal to the capacity of the primal arc it intersects.

Preview of the Algorithm

The algorithm begins by ignoring the budget restriction. All arcs of the primal are initially assigned capacities l_{ij} and the shortest route through the topological dual is determined. The length of the route corresponds to the value of the minimum cut set of the primal when $m_{ij}=l_{ij}$ for all arcs. A check is then made to determine if the interdiction cost for obtaining this minimum cut exceeds the budget constraint. If not, then the problem is solved. If, however, the budget constraint has been exceeded then a reduction in expenditures is required.

The algorithm seeks to "unspend" as carefully as possible so that the amount of flow through the network increases as little as possible. The first step in this unspending operation is to find which arc of the minimum cut set "gives back" the largest amount of expense for the smallest increase in capacity. Unspending takes place until $m_{ij}=u_{ij}$ or the budget constraint is satisfied. If $m_{ij}=u_{ij}$ then the algorithm continues working on the minimum cut set until the budget constraint is satisfied. The final value of that cut set is then determined and retained for later comparisons.

The algorithm looks next for the second shortest route corresponding to the second lowest valued cut set when all arcs have $m_{ij}=l_{ij}$. It repeats the budget check and the unspending process. After the budget is satisfied on this cut set then the cut set value is compared with the final value of the cut set of the "shortest" routes; that cut set having the lower final value is retained and the other is dropped from further consideration.

The process continues with consideration next of the third shortest route or third minimum cut set with all arcs having $m_{ij} = l_{ij}$ and then the fourth and so on. If, at any time, the length of the next shortest route using all l_{ij}'s is greater than the final length of the best previous route, the algorithm terminates. There is no point in continuing the next shortest route investigations since all further routes will have lengths greater than the feasible length of the best previous route.

Feasible Min-Cut Algorithm

1. Construct the topological dual of the network and set all $m_{ij} = l_{ij}$. Set $r = 1$.

2. Determine R_r, the rth shortest loopless route through the dual when $m_{ij} = l_{ij}$, and determine its length L_r^* from

$$L_r^* = \sum_{(i,j)\epsilon R_r} l_{ij}.$$

If $w \geqslant 2$ routes qualify for the rth shortest route because of ties in total length, arbitrarily select one of these routes as the rth, another as the $(r+1)$th, another as the $(r+2)$th, and so on, with the last of the group being designated as the $(r+w-1)$th shortest route.

Compare L_r^* with $L^{(r-1)}$, the length of the shortest feasible route from the set $R_1, R_2, \ldots, R_{r-1}$. (Let $L^{(0)} = \infty$).

(a) If $L_r^* < L^{(r-1)}$ then go to step 3.

(b) If $L_r^* \geqslant L^{(r-1)}$ then terminate the algorithm. The routes $R_r, R_{r+1}, R_{r+2}, \ldots, R_N$ will have feasible lengths which are no shorter than $L^{(r-1)}$ and need not be considered.

3. Compute the interdiction expense, E_r, associated with L_r^* from

$$E_r = \sum_{(i,j)\epsilon R_r} C_{ij}[u_{ij} - l_{ij}].$$

(a) If $E_r \leqslant K$, terminate the algorithm. Route R_r has the minimum feasible length of all routes through the dual.

(b) If $E_r > K$, go to step 4.

4. List the n arcs in R_r in descending order of C_{ij} values; let $C_1(r)$ represent the largest C_{ij} and $C_n(r)$, the lowest. Beginning with $q = 1$ and $L_r = L_r^*$, increase the length of the arc (i, j) corresponding to $C_q(r)$ and the route length L_r by

$$\Delta m_{ij} = \min\left\{u_{ij} - l_{ij}, \frac{E_r - K}{C_{ij}}, L^{(r-1)} - L_r\right\}.$$

Decrease the interdiction expense E_r by $C_{ij}\Delta m_{ij}$.

(a) If $\Delta m_{ij} = u_{ij} - l_{ij}$ increase q by 1; compute Δm_{ij} and the new values of L_r and E_r for the next arc on the C_{ij} list.

(b) If $\Delta m_{ij} = \dfrac{E_r - K}{C_{ij}}$, the interdiction expense for the route is $E_r = K$. If $L_r \leqslant L^{(r-1)}$, set $L^{(r)} = L_r$ and record the current value of q, call it s. Delete the route associated with $L^{(r-1)}$ from further consideration. If $L_r > L^{(r-1)}$, set $L^{(r)} = L^{(r-1)}$ and drop R_r from further consideration. Increase r by 1 and return to step 2.

(c) If $\Delta m_{ij} = L^{(r-1)} - L_r$, the length of route r has been increased to $L^{(r-1)}$, but it is still not feasible since $E > K$. Delete R_r from further consideration, set $L^{(r)} = L^{(r-1)}$, and return to step 2.

If there is a tie between $u_{ij}-l_{ij}$ or $L^{(r-1)}-L_r$ and $\dfrac{E_r-K}{C_{ij}}$ for value of Δm_{ij}, apply part (b) above. If there is a tie between $u_{ij}-l_{ij}$ and $L^{(r-1)}-L_r$, apply part (c).

Optimal Allocation

The value of $L^{(r)}$ at the termination of the algorithm is the minimum value of all the feasible cut sets. This is the minimum achievable network capacity. The interdiction effort is assigned to the arcs of the primal which are "cut" by the feasible route R_p of the topological dual associated with the value of $L^{(r)}$. The optimal number of sorties to allocate is

$$n_{ij}=C_{ij}[u_{ij}-l_{ij}]$$

for the arcs of the primal cut by the dual arcs of R_p associated with $C_{s+1}(p)$, $C_{s+2}(p)$, . . ., $C_n(p)$ where s is the index from the C_{ij} list of the first arc on R_p having $\Delta m_{ij}>0$. For the arc (i,j) associated with $C_s(p)$:

$$n_{ij}=K-\sum_{C_{s+1}(p)}^{C_n(p)} n_{ij}.$$

Finally, $n_{ij}=0$ for all other arcs of the primal network.

EXAMPLE: Figure 1 presents the network information for the example. The value of K will be 5. Node 1 is the source and node 5 is the sink. The numbers on each arc represent l_{ij}, u_{ij}; C_{ij}.

The topological dual is formed as shown by the dashed lines in Figure 1. The artifical arc added to the primal for constructing the dual is arc (5, 1). The completed topological dual is shown in Figure 2; the numbers on the arcs represent the upper and lower bounds on arc length and the unit costs for shortening them. These numbers correspond directly to the numbers on the arcs of the primal cut by the dual arcs. The source and sink of the dual are nodes A and D, respectively.

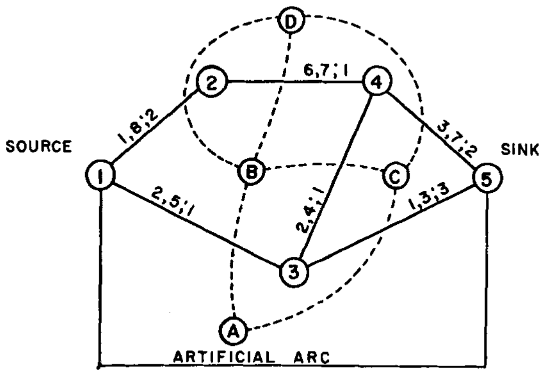

FIGURE 1. A supply network

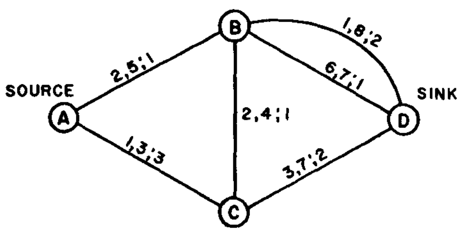

FIGURE 2. The topological dual of the network

When $m_{ij} = l_{ij}$ on all of the arcs of the dual the complete set of loopless routes from source to sink with associated lengths L_r^* can be obtained by inspection. It is:

R_1: (AB, BD1)	$L_1^* = 3$
R_2 : (AC, CD)	$L_2^* = 4$
R_3 : (AC, CB, BD1)	$L_3^* = 4$
R_4 : (AB, BC, CD)	$L_4^* = 7$
R_5 : (AB, BD2)	$L_5^* = 8$
R_6 : (AC, CB, BD2)	$L_6^* = 9.$

The designation BD1 is associated with the upper BD arc in Figure 2 and BD2 is associated with lower. Although the algorithm would not evaluate all routes R_1 through R_6 and their associated L_r^* values they are presented for the sake of discussion.

The algorithm begins by finding R_1 and computing $L_1^* = 3$. $L^{(0)} = \infty$ is set so that $L_1^* < L^{(0)}$. Because $E_1 = 17 > K$, the cost coefficients for R_1 are ranked, $C_1(1) = 2$ (for arc BD1) and $C_2(1) = 1$ (for arc AB). The evaluation of Δm_{BD1} results in

$$\Delta m_{\text{BD1}} = \frac{E_1 - K}{C_{\text{BD1}}} = 6,$$

$L_1 = 9$, and $E_1 = 5 = K$. The analysis of R_1 is complete because $E_1 = K$, therefore $L^{(1)} = L_1 = 9$.

After finding R_2, the value L_2^* is computed. Because $L_2^* = 4 < L^{(1)}$, the value of E_2 is next determined. $E_2 = 14 > K$ so the cost coefficients for R_2 must be ranked. $C_1(2) = 3$ (for arc AC) and $\Delta m_{\text{AC}} = u_{\text{AC}} - l_{\text{AC}} = 2$ resulting in $L_2 = 6$ and $E_2 = 8$. Next $\Delta m_{\text{CD}} = \dfrac{E_2 - K}{C_{\text{CD}}} = 3/2$ so $L_2 = 7\frac{1}{2}$ and $E_2 = 5 = K$, completing the analysis of R_2.

Because $L_2 < L^{(1)}$ we drop R_1 from further consideration and set $L^{(2)} = L_2 = 7\frac{1}{2}$.

R_3 is next on the list. $L_3^* < L^{(2)}$ so E_3 is determined. $E_3 = 22 > K$ and Δm_{AC} must then be calculated. We get $\Delta m_{\text{AC}} = u_{\text{AC}} - l_{\text{AC}} = 2$ resulting in $L_3 = 6$ and $E_3 = 16$. Next, $\Delta m_{\text{BD1}} = L^{(2)} - L_3 = 3/2$ and R_3 can be disregarded. Set $L^{(3)} = L^{(2)} = 7\frac{1}{2}$.

Route R_4 has $L_4^* = 7 < L^{(3)}$ and $E_4 = 13$. Then $\Delta m_{\text{CD}} = L^{(3)} - L_4 = 1/2$ and we can disregard R_4. Set $L^{(4)} = L^{(3)} = 7\frac{1}{2}$.

Because R_5 has $L_5^* = 8 > L^{(4)}$ the algorithm terminates.

The dual route which is used to determine the optimal allocation of interdiction effort is R_2. $L_2 = 7\frac{1}{2}$ is the value of the minimum cut of the primal network after optimal interdiction. Arc AC has length $m_{\text{AC}} = u_{\text{AC}} = 3$ and arc CD has a length $m_{\text{CD}} = 4\frac{1}{2} < u_{\text{CD}}$. Therefore arc (3, 5) of the primal has a final capacity of $m_{35} = u_{35} = 3$ and arc (4, 5) of the primal has a final capacity of $m_{45} = 4\frac{1}{2}$. The entire budget $K = 5$ is allocated to interdiction of arc (4, 5). This optimal interdiction gives a maximum possible flow through the network of $7\frac{1}{2}$.

An *r*th Shortest Route Algorithm

An algorithm for finding the *r*th shortest loopless route through the dual network is a necessary part of step 2 of the Feasible Min-Cut algorithm for large problems. Such an algorithm can be derived by minor modifications to the "*N* best loopless paths" algorithm of Clarke, Krikorian, and Rausen [2] (their algorithm will be referred to as the CKR algorithm from this point on). In seeking the *N* best loopless paths the CKR algorithm concentrates on paths which have at most one loop. The procedure begins with the determination of an initial set S of N loopless routes along with a set T of routes having one loop, but lengths less than the longest of the N routes of S. Special deviations, called "detours," from routes in the set T are then examined to see if any loopless route arises which is shorter in length than the longest of set S. If so, then this route replaces the longer one in S. When the elements of sets S and T cease changing the algorithm terminates.

The modification for converting this procedure to an rth shortest route type is quite simple. Use the CKR algorithm to find an initial set of $N \geq 1$ best loopless routes. If, during the course of applying the Feasible Min-Cut algorithm additional routes beyond N are needed, use the existing N routes to initiate the construction of the new set S. The new set S is initially established when a specified number of loopless routes, $K(\geq 1)$, has been added to S. Those detours of routes in new S having loops, but total lengths less than the maximum from S form the new set T. The CKR algorithm is then applied to find the final set of $N+K$ best loopless routes.

If more than $N+K$ routes are needed after returning to the Feasible Min-Cut algorithm then another set of K additional routes can be added in the same way as the first K. The second new set S would be initiated with the existing $N+K$ best loopless routes.

The values of N and K are a matter of personal choice. The use of $K=1$ does not however seem very efficient because of the possibility of multiple routes of the same length. With $K>1$ such ties become more quickly apparent. In any case, a complete list of all routes of a particular length should be evaluated before returning to the Feasible Min-Cut algorithm. For example, if there are three shortest routes through the network and $N=2$ was used then an additional set of $K \geq 2$ routes should be evaluated to pick up the third route and to show that there is only one more shortest route prior to going to step 3 of the Feasible Min-Cut algorithm.

Modifications when all $l_{ij}=0$

The Feasible Min-Cut algorithm was designed for problems where at least one arc has $l_{ij}>0$. The reason for this was that in most real-world interdiction problems it would be virtually impossible to reduce an arc's capacity to zero for any extended period of time [3, 6]. Often hand-carrying of supplies can begin immediately after an aerial or ground attack. If one considers l_{ij} to represent the average 24 hour minimum capacity then hand-carrying and minor repairs would definitely result in $l_{ij}>0$.

If the Feasible Min-Cut algorithm is applied to a network having all $l_{ij}=0$ it would evaluate the feasible length of all loopless routes through the dual. The following modifications in steps 1 and 2 of the algorithm are suggested as a means of possibly avoiding this complete evaluation. Step 3 would be by-passed completely.

1. Construct the topological dual of the network and set all $m_{ij}=u_{ij}$. Set $r=1$.

2. Determine R_r, the rth shortest loopless route through the dual when $m_{ij}=u_{ij}$. Then set $m_{ij}=0$ for all arcs on this route and determine E_r from

$$E_r = \sum_{(i,j)\epsilon R_r} C_{ij} u_{ij}.$$

(a) If $E_r \leq K$, terminate the algorithm. Route R_r has a minimum feasible length of zero and $n_{ij}=C_{ij}u_{ij}$ for all arcs on R_r.

(b) If $E_r > K$, go to step 4.

Comments

The algorithm terminates in a finite number of steps since the number of loopless routes through the dual network is finite for finite networks and each route is examined only once.

If all l_{ij}, u_{ij}; C_{ij}, as well as K are integer valued then n_{ij} will be integer also. If any of these parameters is not integer then there is no guarantee of an integer solution. If a problem involves allocating sorties then integer solutions should be sought after the Feasible Min-Cut algorithm is completed. If, however, the problem involves allocating, say, tons of bombs, then noninteger results might be quite reasonable.

Extensions

The law of diminishing returns suggests that actual interdiction costs for an arc (i, j) may follow a curve of the type shown in figure 3. The Feasible Min-Cut algorithm can solve problems having this type of nonlinear cost function if the function is replaced by a piecewise linear approximation such as that shown by the dashed lines in Figure 3. This linear approximation can be created in the primal network by replacing arc (i, j) by three arcs having l_{ij}, u_{ij}, and C_{ij} values as shown in Figure 4. The construction of the topological dual will then require that a node be placed in each mesh of Figure 4.

A further extension of the interdiction problem with nonlinear costs has been made by Nugent [7]. He considers an exponential cost function in continuous form and presents an algorithm similar to the Feasible Min-Cut algorithm for solving the problem.

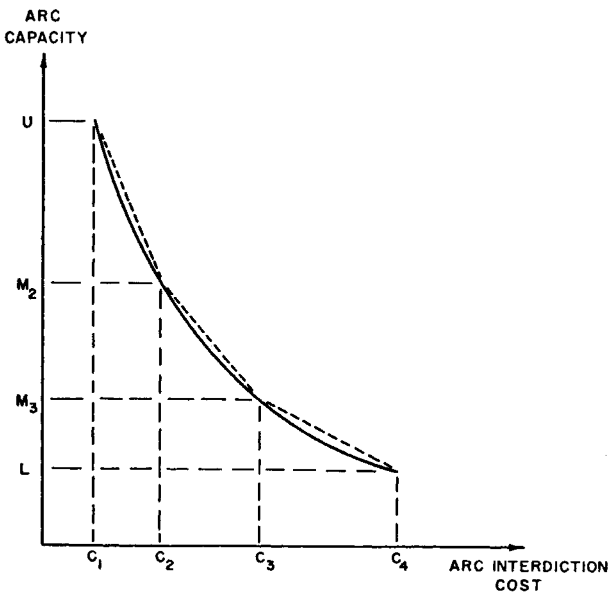

FIGURE 3. Arc capacity as a function of interdiction cost under the law of diminishing returns

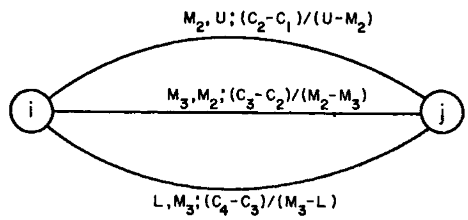

FIGURE 4. Replacement of arc (i, j) for the linear approximation to Fig. 3

REFERENCES

[1] Bellmore, M., J. J. Greenberg, and J. J. Jarvis, "Optimal Attack of a Communications Network," Paper WA 2.4, presented at the 32d National ORSA Meeting, Chicago, November 1967.

[2] Clarke, S., A. Krikorian, and J. Rausen, "Computing the N Best Loopless Paths in a Network," J. SIAM **11**, 1096–1102 (1963).

[3] Durbin, E. P., "An Interdiction Model of Highway Transportation," The RAND Corporation, Rpt. RM–4945–PR (1966).

[4] Ford, L. R. and D. R. Fulkerson, "Maximal Flow through a Network," Canadian J. Math. **8**, 399–404, 1956

[5] Ford, L. R. and D. R. Fulkerson, *Flows in Network* (Princeton Univ. Press, Princeton, N.J., 1962).

[6] Futrell, R. F., *The United States Air Force in Korea, 1950–1953* (Duell, Sloan, and Pearce, New York, 1961).

[7] Nugent, R. O., "Optimum Allocation of Air Strikes Against a Transportation Network for an Exponential Damage Function," Unpublished Masters' Thesis, Naval Postgraduate School, 1969.

[8] Thomas, C. J., "Simple Models Useful in Allocating Aerial Interdiction Effort," Paper WP4.1, 34th National ORSA Meeting, Philadelphia, Nov. 1968.

[9] Wollmer, R. D., "Removing Arcs From a Network," Operations Research **12**, 934–940 (1964).